Events around the Black Hole

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Keywords: Black hole, General relativity, Event horizon, Light cone, Penrose diagram, Spaghettification

Abstract: In this paper, we investigate a portion of events around a black hole in the view of General Relativity. These events are briefly classified as the “darkness” of black hole and the “danger” around black hole. In other words, we explain why light cannot escape from black holes and why black holes would tear up everything nearby. In the literature review section, we review the history of general relativity, from Maxwell equations to special relativity. In the introduction section, we briefly introduce the basic concept of Riemannian geometry and derive the Einstein equations for gravitation which are the basic of general relativity. The Schwarzschild metric, an explicit solution to Einstein equations under certain assumption, is also discussed. We elaborate in the third and fourth section some cosmological consequences of this solution to answer previous questions. In the third section, we introduce the “darkness” of black hole in various aspects, like exchanging time and radius coordinates, photon sphere and Penrose diagram, etc. In the fourth section, we discuss why black hole is so dangerous to astronauts and celestial bodies in the universes by deriving geodesic deviation equation and tidal force. In the last section, we conclude and summarize our discussions.

1. Introduction

The spacetime indices are denoted by Greek letters like μ and ν. Since the interested dimensions in the general relativity include three spatial axes and one time axis, there are four possible values for the spacetime indices[1]:

$\mu, \nu = 0, 1, 2, 3.$

The value 0 denotes the time dimension and other values denote spatial dimensions.

For simplicity of discussion, we use various derivative notations[2]:

for ordinary derivative,

$$\frac{\partial A_\mu}{\partial x^\nu} = \partial_\nu A_\mu = A_{\mu,\nu},$$

for covariant derivative,

$$\frac{DA_\mu}{Dx^\nu} = \nabla_\nu A_\mu = A_{\mu;\nu}.$$

The expression of matrixes or tensors by their general components is applied:[3]

$$T^{\mu\nu} = \left(T^{\mu\nu}\right)_{\substack{\mu=0,1,2,3\\ \nu=0,1,2,3}},$$

$$x^\mu = \left(x^0, x^1, x^2, x^3\right).$$

By Einstein summation convention, repeated dummy indices are used during summation[4]. For example,

$$A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3,$$

$$\partial_\mu A^\mu = \partial_0 A^0 + \partial_1 A^1 + \partial_2 A^2 + \partial_3 A^3 = \mathrm{div}(A^\mu),$$

$$T^\mu{}_\mu = T^0{}_0 + T^1{}_1 + T^2{}_2 + T^3{}_3 = \mathrm{Tr}(T^\mu{}_\nu).$$

Throughout this paper, we set c=1 for the speed of light for convenience of discussions[5].

2. Literature Review

In 1873, J. Clerk Maxwell established the Maxwell equations, which indicate that the light speed is a constant [6]. However, the speed of light is variant under Galilean transformation in the frame of classical mechanics. For example, when you are moving towards the light source, the light propagates faster in your frame of reference. To deal with this contradiction, scholars proposed a hypothesis that there is a medium called 'ether' and the constant speed of light is relative to such medium.

In 1887, Michelson and Morley conducted an experiment in order to find ether [7]. But the experimental result denies the existence of ether. This result greatly shakes the foundation of classical mechanics. The establishment of a new theory is urgent in order to remove such dilemma, which facilitates the birth of special relativity.

Albert Einstein proposed the special theory of relativity in 1905 based on two assumptions. [8]. Firstly, the laws of physics are invariant in all inertial frames of reference. Secondly, the speed of light in vacuum is invariant for all observers [9]. In the view of special relativity, the conventional three-dimension space evolves to the four-dimension spacetime. Phenomena such as time dilation and length contraction are direct consequences of special relativity.

In 1915, Einstein improved his theory and proposed general relativity [10], in which he abandoned the obscure definition of inertial frames and generalized the principle of covariance. That is, the form of every physical theory is invariant with respect to any reference frame. Einstein field equations tell us matter curves spacetime. General relativity perfectly explains phenomena such as the precession of Mercury, deflection of light beam and spectrum frequency shift.

Schwarzschild solved Einstein field equations in vacuum spacetime [11]. The Schwarzschild's solution reveals a special structure of spacetime, the Black Hole.

In the following parts of this paper, we are going to introduce such black hole and discuss consequence caused by this special metric.

Let $g_{\mu\nu}$ be the metric tensor of spacetime. The line element of spacetime is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

where the invariant parameter $d\tau = ds$ is also called *proper time* which describe the physical time that the test particle experiences.

In the flat-spacetime case,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (2)$$

where $\eta_{\mu\nu}$ is called Minkowski metric tensor.

To describe the curvature of the spacetime, we firstly introduce

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right), \quad (3)$$

called the *Christoffel symbols of the first kind*, which describes the motion of affine frame.

Let $g^{\mu\nu}$ be the inverse of $g_{\mu\nu}$, i.e.,

$$g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^{\ \nu}. \quad (4)$$

Then we can lift or lower the indices with the help of these metrics, like

$$\Gamma^\mu_{\ \alpha\beta} = g^{\mu\nu} \Gamma_{\nu\alpha\beta}, \quad (5)$$

which is called the *Christoffel symbols of the second kind.*

Here we can also introduce the definition of covariant derivative:
for the covariant tensor A_μ

$$A_{\mu;\alpha} = A_{\mu,\alpha} - \Gamma^{\beta}{}_{\mu\alpha} A_\beta, \quad (6)$$

for the contravariant tensor A^μ

$$A^{\mu}{}_{;\alpha} = A^{\mu}{}_{,\alpha} + \Gamma^{\mu}{}_{\beta\alpha} A^\beta. \quad (7)$$

We could now define the "straight line" on a curved spacetime. Let $x^\alpha(\lambda)$ be the coordinate equation of a "straight line", where λ is its affine parameter. "Straight" means the 4-acceleration

$$\frac{D^2 x^\alpha}{d\lambda^2} = 0. \quad (8)$$

By the definition of contravariant tensor Eq.(8), we get the *geodesic equation*,

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^{\alpha}{}_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0. \quad (9)$$

Now we can introduce a variety of curvature tensors:
Riemann curvature tensor $R^{\nu}{}_{\mu\alpha\beta}$

$$R^{\nu}{}_{\mu\alpha\beta} = \Gamma^{\nu}{}_{\mu\beta,\alpha} - \Gamma^{\nu}{}_{\mu\alpha,\beta} + \Gamma^{\gamma}{}_{\mu\beta}\Gamma^{\nu}{}_{\gamma\alpha} - \Gamma^{\gamma}{}_{\mu\alpha}\Gamma^{\nu}{}_{\gamma\beta}. \quad (10)$$

Ricci tensor $R_{\mu\nu}$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}. \quad (11)$$

Scalar curvature R

$$R = g^{\mu\nu} R_{\mu\nu}. \quad (12)$$

Einstein tensor $G^{\mu\nu}$

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R. \quad (13)$$

"*Matter curves spacetime*" is the basic idea that indicated by Einstein's general relativity theory. In the language of mathematics, he wanted to find some tensor $Q^{\mu\nu}$ such that

$$Q^{\mu\nu} = -\kappa T^{\mu\nu}, \quad (14)$$

where $T^{\mu\nu}$ is called *energy-momentum tensor* resembling matters in the universe and satisfy the *pointwise conservation law* implied by the *Noether theorem*. That is

$$T^{\mu\nu}{}_{;\nu} = 0. \quad (15)$$

Therefore, such tensor $Q^{\mu\nu}$ also needs to satisfy

$$Q^{\mu\nu}{}_{;\nu} = 0. \quad (16)$$

Though both the Einstein tensor $G^{\mu\nu}$ and the metric tensor $g_{\mu\nu}$ could be the candidates, $g_{\mu\nu}$ is always non-degenerate but $T^{\mu\nu}$ could be degenerate in the empty space. Einstein finally established his *Einstein field equations without cosmic constant*,

$$G^{\mu\nu} = -8\pi G T^{\mu\nu}, \quad (17)$$

where $\kappa=8\pi G$ is determined by the weak gravitational estimation and G is the gravitational constant.

Though it is hard to find an explicit solution to the Einstein equation in the general case, we can find some solution under certain condition.

Now assume the spacetime is spherical symmetry and static

$$\partial_t g_{\mu\nu} = 0. \quad (18)$$

Changing the Cartesian coordinates ($x^0=t, x^1, x^2, x^3$) to the spherical coordinates (t, r, θ, ϕ),

$$\begin{cases} x^1 = r\sin\theta\cos\phi, \\ x^2 = r\sin\theta\sin\phi, \\ x^3 = r\cos\theta, \end{cases} \quad (19)$$

the metric element takes the form:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (20)$$

by the spherical symmetry condition.

Calculate the Ricci curvatures

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \quad (21)$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \quad (22)$$

$$R_{22} = -1 + \frac{1}{B} + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \quad (23)$$

$$R_{33} = \sin^2\theta R_{22}. \quad (24)$$

where $A' = \frac{dA}{dr}$ and $B' = \frac{dB}{dr}$.

Consider a vacuum spacetime outside a centrally localized mass M, where

$$T = T_{\mu\nu} = R = R_{\mu\nu} = 0. \quad (25)$$

Combining Eq.(21) and (22), we get $(AB)'=0$, i.e., AB is a constant.

The asymptotic limit, $A=B=AB=1$ as $r\rightarrow\infty$, indicates the value of AB is one and

$$A = \frac{1}{B}, \text{ for any } r>0. \quad (26)$$

Using this in Eq.(23), we get

$$rA' + A = 1. \quad (27)$$

The integration of Eq. (27) yields

$$A(r) = 1 + \frac{C}{r}. \quad (28)$$

for some constant C. Also using asymptotic condition that general relativity will recover *Newton's law of gravitation*, the constant C is determined as follow:

$$C = -2GM. \quad (29)$$

Finally, we arrive at the Schwarzschild solution to Einstein equation and the metric can be represented by

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (30)$$

which is valid outside the central mass M.

Notice that this solution brings a coordinate singularity at

$$r = r_s = 2GM \quad (31)$$

where r_s is called *Schwarzschild radius*.

In the general case, r_s is less than the radius R of the central mass, where the solution is no longer valid. For example, the Schwarzschild radius of the earth and the sun are 9 mm and 3 km while the radius of the earth and the sun are 6371 km and 696300 km, respectively. The event horizons of both are buried deep beneath their surfaces. Hence, neither of them is a black hole. There is a supermassive black hole at the center of our Milky Way galaxy known as Sagittarius A^*, the Schwarzschild radius of which is 2×10^7 km. Its photo is displayed in the press conference on May 12, 2022 taken by Event Horizon Telescope (EHT).

When the central mass is highly concentrated and $r_s>R$, the main topic of this paper, the black hole, occurs. The sphere surface $r=r_s$ is called *event horizon*. In the following part of this paper, we will discuss why black hole is "black" and why black hole is "dangerous".

3. Why Black Hole is "Black"?

The foundation to answer this question is that the detection of light is the key that forms our visual perception. Light is generated from the stars in our universe and finally arrives at our eyes so we can see stars far away. As for the black hole, light will never pass through the event horizon due to its special metric, which makes the black hole invisible to us. The following content gives a deeper discussion for the light around the black hole.

3.1 Exchanging Role between Time and Radius

We first answer this question using the most intuitive and naïve perspective. Firstly, recall the Minkowski metric Eq.(2), which consists with three space dimensions and one time dimension. The only one metric component of time η_{00} is positive and three space components are negative. So do the Schwarzschild metric outside the event horizon. For photons in the event horizon $r<r_s$, the signs of the factors of dt^2 and dr^2 are converted. It seems that coordinate time and radius exchange their role when passing through the event horizon. It should be noticed that it is hard for time to change its direction since it is 1-dimensional. Changing the direction of radial movement will also be difficult for photons inside the event horizon. Therefore, photons can do nothing but falling towards the center of black hole.

However, general relativity does not rule out the possibility that time reverses its direction. More aspects are needed to hold our view.

3.2 Lightlike Propagation and Light Cone

Consider light propagating along the radial direction, which is the most possible direction for light to get out of the black hole in the view of mechanics. That is,

$$ds^2 = 0, d\theta = d\phi = 0. \quad (32)$$

For the flat metric

$$ds^2 = dt^2 - dr^2. \quad (33)$$

Hence,

$$r = t, \quad (34)$$

In which the integration constant is neglected.

As shown in Figure 1, light propagates in a flat spacetime with the speed 1, the usual speed of light in our content

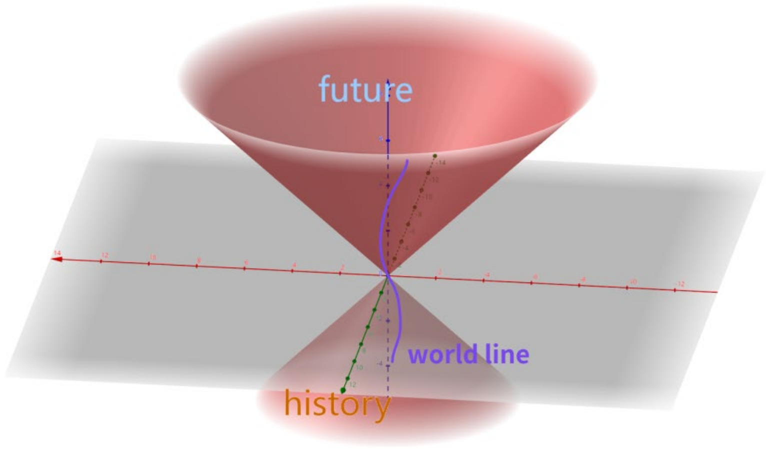

Fig.1 Diagram of the Light Cone

Any world line must be bounded in the light cone since nothing can be faster than light.

As for the Schwarzschild metric, such photon satisfies

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0. \quad (35)$$

i.e.,

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2GM}{r}\right)^2. \quad (36)$$

Now we need to talk about the propagation of photon in different cases.

(i)For light outside the event horizon, $r>r_s=2GM$, solving Eq.(2.5) yields

$$r + r_s \ln(r - r_s) = \pm t + r_0 + r_s \ln(r_0 - r_s), \quad t \geq 0. \quad (37)$$

where $r_0=r(0)$. Hence, t tends to infinity as r tends to r_s^+, which means it would take infinite coordinate time for the photon to get to the event horizon.

(ii)For light inside the event horizon, $r<r_s=2GM$, solving Eq.(2.5) yields

$$r + r_s \ln(r_s - r) = \pm t + r_0 + r_s \ln(r_s - r_0), \quad t \geq 0. \quad (38)$$

Hence, t tends to infinity as r tends to r_s^-, which means the photon will never reach the event horizon in finite coordinate time. Also, we can notice that it only takes finite time for the photon to fall into the spacetime singularity $r=0$.

Therefore, if we see something falling onto the event horizon, even we cannot see the event horizon, it would "freeze" on the event horizon surface because its next move happens in the infinite future for us. Also, if something inside emits light ray, the light would "freeze" on the event horizon surface thus unreachable. As a result, the black hole is "black" to us. It can be seen that, the culprit is the factor before dt^2, g_{00}, who tends to zero when getting close to the event horizon.

Another consequence g_{00} cause is called redshift. The period of the wave is defined as the time intersection of two adjacent wave crests as. On the event horizon, time is frozen so we can never get two wave crests of light, which means the period of light wave becomes infinity and the frequency of light tends to zero. Light on the event horizon becomes infinitely far infrared. In this case, such event horizon is also called the surface of infinite redshift.

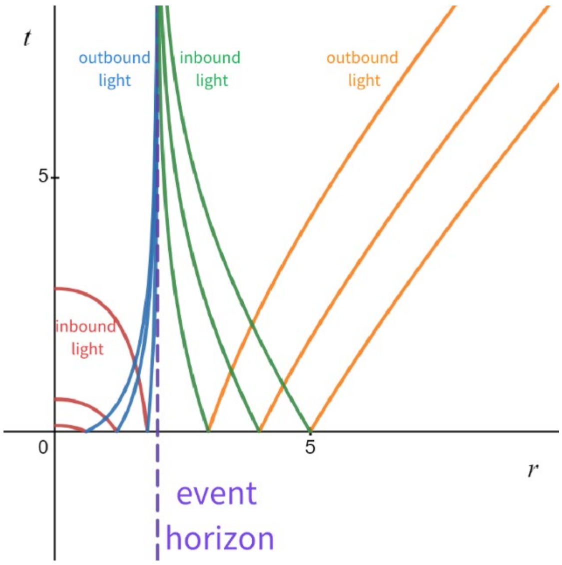

Fig.2 Lightlike Motion in Radial Direction

It should be noticed that the "infinite time" discussed in the preceding content is the coordinate time, or say the time that observer far away experiences. The proper time τ is the physical time that space traveler really experiences and is invariant with respect to coordinate transformation. Therefore, even though we see space travelers frozen on the event horizon, passing through the event horizon would not be hard when neglecting other phenomena. Therefore, the event horizon is not an essential spacetime singularity. Such metric singularity occurs because we choose wrong coordinates just like we cannot talk about the north and south pole (θ=0 *or* π) using the standard sphere metric,

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (39)$$

where the factor before $d\phi^2$ is also zero but such flat spacetime will not cause any trouble when using Cartesian coordinates.

3.3 Equation of General Motion and Light Sphere

In the previous sections, we discussed the motion of light in radial direction. We are going to discuss more general equations of motion. In the general case, we may use the static spherical metric Eq.(40)

$$d\tau^2 = ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (40)$$

and geodesic equation which describes free falling Eq.(41),

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = 0. \quad (41)$$

Since the metric is spherically symmetric, we assume all motions are in the equator plane, $\theta = {}^{\pi}\!/_{2}$, by rotating transition of coordinate. In this case, Eq.(42) and Eq.(43) become,

$$\left(\frac{d\tau}{dp}\right)^2 = E, \quad (42)$$

$$r^2\frac{d\phi}{dp} = J, \quad (43)$$

$$\frac{dt}{dp} = \frac{1}{B}, \quad (44)$$

$$A\left(\frac{dr}{dp}\right)^2+\frac{J^2}{r^2}-\frac{1}{B}=-E, \quad (45)$$

where J and E are integration constants. Eq.(44) and (45) stand for angular momentum conservation and energy conservation (4). Generally speaking, observer uses coordinate time to describe the motion of observed object. We may use Eq.(46) to replace p with t, then

$$d\tau^2=EB^2dt^2, \quad (46)$$

$$\frac{A}{B^2}\left(\frac{dr}{dt}\right)^2+\frac{J^2}{r^2}-\frac{1}{B}=-E, \quad (47)$$

$$r^2\frac{d\phi}{dt}=JB. \quad (48)$$

where $A=A(r)$ and $B=B(r)$. These equations govern the motions on the equator plane.
Insert Eq.(48) to Eq.(47) using $\frac{dr}{dt}=\frac{dr}{d\phi}\frac{d\phi}{dt}$ and integrate,

$$\phi=\pm\int\frac{\sqrt{A}dr}{r^2\sqrt{\frac{1}{J^2B}-\frac{E}{J^2}-r^2}}, \quad (49)$$

which is the orbit equation on the equator plane.
Next, we are going to find a circular orbit. Let the radius of this orbit be R, then

$$\frac{d^2r}{dp^2}=\frac{dr}{dp}=0. \quad (50)$$

Inserting Eq.(50) to Eq.(40) when $\mu=1$, we get

$$0=-\frac{R}{A(R)}\left(\frac{d\phi}{dp}\right)^2+\frac{B'(R)}{2A(R)}\left(\frac{dt}{dp}\right)^2. \quad (51)$$

For the photons, $ds^2=0$

$$0=ds^2=A(r)dt^2-r^2d\phi^2, \quad (52)$$

i.e.,

$$\left(\frac{d\phi}{dt}\right)^2=\frac{A(R)}{R^2}. \quad (53)$$

Combined with Eq.(44),

$$0=-\frac{R}{A(R)}\frac{A(R)}{R^2}\frac{1}{B(R)^2}+\frac{B'(R)}{2A(R)}\frac{1}{B(R)^2}. \quad (54)$$

For Schwarzschild black hole,

$$A(R)=1-\frac{r_s}{R}, \quad B(R)=\left(1-\frac{r_s}{R}\right)^{-1}. \quad (55)$$

Solve Eq.(54) and finally we get

$$R=\frac{3}{2}r_s=3GM, \quad (56)$$

which is the radius of the photon sphere. Different from the event horizon, where any photon will "freeze", any photon that crosses the photon sphere outward will go to infinity and any photon that crosses the photon sphere inward will fall into the spacetime singularity.

3.4 Penrose Graph

Another intuitional way to understand the "blackness" of the black hole is the Penrose graph. The Penrose Graph brings the infinite spacetime into a bounded region without changing the light cone as in Figure 1 by using conformal transformation.

For example, consider the flat (1,1)-Minkowski metric,

$$ds^2 = dt^2 - dr^2. \quad (57)$$

Introduce

$$\begin{cases} U = t + r \\ V = t - r \end{cases}. \quad (58)$$

To bring infinity to finite,

$$\begin{cases} U = \tan u, \\ V = \tan v. \end{cases} \quad (59)$$

New time parameter t' and new radius parameter r' are defined as,

$$\begin{cases} t' = \dfrac{u+v}{2}, \\ r' = \dfrac{u-v}{2}. \end{cases} \quad (60)$$

Then $t' \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $r' \in (0, \frac{\pi}{2})$. Now we get the Penrose diagram for the (1,1) Minkowski space as shown in Figure 3.

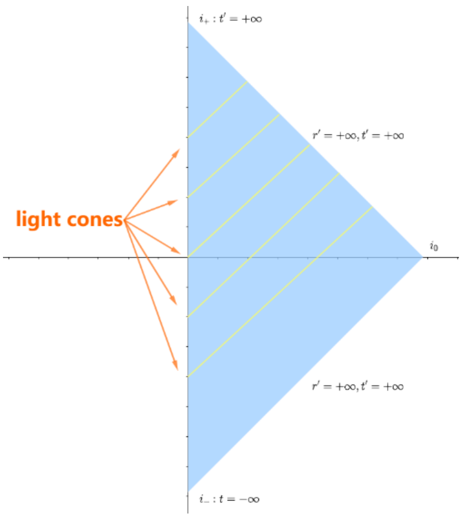

Fig.3 Penrose Diagram for Minkowski Space

For the Schwarzschild metric, the *Kruskal-Szekeres coordinates* are defined as,

$$\xi = |r - r_s|^{\frac{1}{2}} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right), \quad (61)$$

$$\eta = |r - r_s|^{\frac{1}{2}} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right), \quad (62)$$

for $r \neq r_S$.

The metric is updated to

$$ds^2 = \pm \frac{4r_s^2}{r} e^{-\frac{r}{r_s}} (d\xi^2 - d\eta^2) - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (63)$$

Repeat the procedure from Eq. (61) to Eq.(63). We can get the Penrose diagram for the Schwarzschild spacetime.

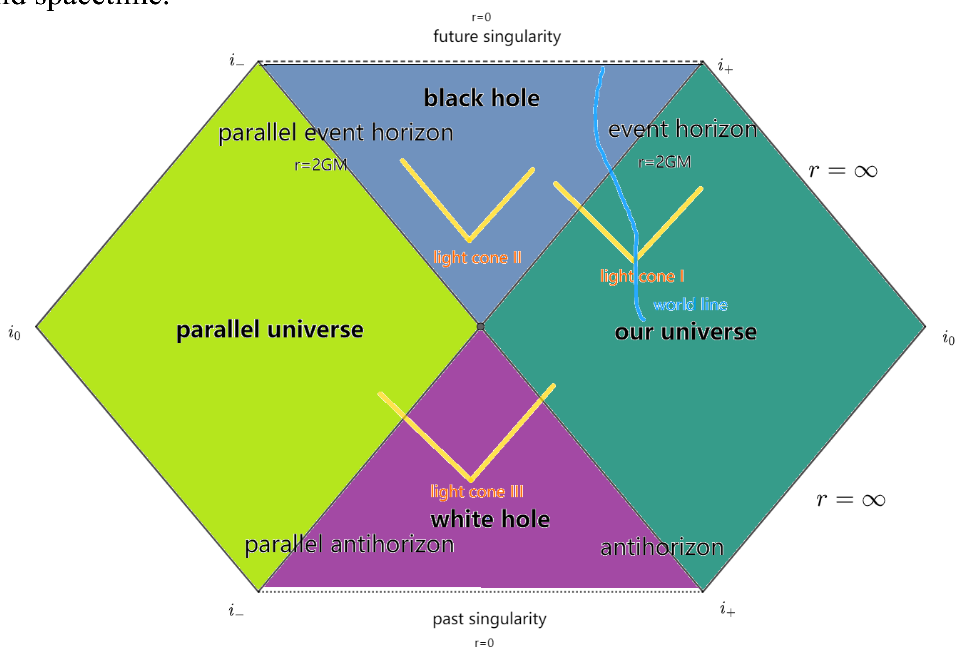

Fig.4 Penrose Diagram for the Schwarzschild Black Hole

As we can see, the light cone I starts from "our universe", which partially intersects with the black hole. Hence, timelike particles may or may not fall into the black hole. The blue timelike world line is an unlucky particle that falls into the black hole and finally hits onto the singularity r=0. Light cone II starts from the black hole and is completely "swallowed" by the black hole, which means the future of light cone II belongs to the black hole. By time reversal, we can get the "parallel universe" and "white hole". Anything in the parallel universe behaves the same as our universe but anything in the white hole is contrary to the black hole. Particles will emerge from the antihorizon or parallel antihorizon and appear in "our universe" or "parallel universe".

In the previous sections, we discussed why black hole is "black"-- why we cannot see anything including light inside the event horizon. However, even though we are trapped by a black hole, we are just in free fall. If that is the case, we feel nothing except weightlessness and nothing would happen before we knock into the black hole. In this section we are going to discuss why black hole is so dangerous. In another word, why black hole would destroy anything nearby?

There will still be other effects causing black hole dangerous like frictional force, X-rays with high energy or Hawking radiation that we are not going to talk about since this paper only covers the effect of general relativity.

3.5 Spaghettification

Assume that a set of test particles start from the same spherical surface with the same radial velocity. Then, they must end at the same spherical surface since the spacetime is spherical symmetry, or say "isotropy". Notice that the area of a spherical surface is

$$S = 4\pi r^2, \quad (64)$$

which shrinks to zero as the particles falling to the black hole. Therefore, this set becomes denser and denser and finally infinite density.

Consider an astronaut to be the set of test particles free-falling close enough to the singularity,

$r \to 0$. As shown in Figure 2, the coordinate time t of all free-falling particles tends to constant. Let constant Δt be the coordinate time interval between the astronaut's head and feet. Recall that the coordinate time t and radius r exchange their roles inside the event horizon. The length l of the astronaut can be calculated as

$$l = \int_{feet}^{head} \sqrt{g_{00}}\, dt \doteq \left(1 - \frac{r_s}{r}\right)^{\frac{1}{2}} \Delta t \sim r^{-\frac{1}{2}}, \qquad as \quad r \to 0. \quad (65)$$

Also, the θ and ϕ components of each particle remain constant. Let $\Delta\theta$ and $\Delta\phi$ be the maximal latitude and longitude interval. The cross-sectional area of this astronaut is,

$$A = \iint \sqrt{g_{22} g_{33}}\, d\theta d\phi = r^2 \Delta \cos\theta \Delta\phi \sim r^2, \qquad as \quad r \to 0, \quad (66)$$

which is the same as Eq.(3.1) shows.

The astronaut can also use the proper time to record this free-falling process. Let $\Delta\tau$ be the proper time interval that it takes to get to the singularity. By our assumption, for each particle, θ, ϕ remains constant and t tends to be constant. Then,

$$d\tau^2 \doteq g_{11} dr^2, \quad (67)$$

i.e.,

$$\Delta\tau = \int_{process} \left(1 - \frac{r_s}{r}\right)^{-\frac{1}{2}} dr \sim r^{\frac{3}{2}} \to 0, \qquad as \quad r \to 0. \quad (68)$$

Hence, Eq.(3.2) and Eq.(3.3) tell us,

$$l \sim \Delta\tau^{-\frac{1}{3}}, \qquad A \sim \Delta\tau^{\frac{4}{3}}, \qquad as \quad \Delta\tau \to 0. \quad (69)$$

From Eq.(3.2) and Eq.(3.3), we can see that the length of the astronaut becomes longer while the width of the astronaut becomes smaller and smaller like a spaghetti. Such phenomenon is called *spaghettification* (3).

3.6 Tidal Force and Geodesic Deviation

By Newton's law of gravitation, we can explain tides on the earth easily. Consider the gravitational field of the moon,

$$V = \frac{GM}{r}, \quad (70)$$

as well as the Newton's law of motion,

$$\vec{a} = -\nabla V, \quad (71)$$

where M is the mess of the moon, $\vec{a}$ is the acceleration.

Let R be the distance between the center of earth and moon and r_0 be the radius of the earth. The acceleration of seawater closer to the moon $\vec{a}_1$ and the acceleration further from the moon $\vec{a}_2$.is different from the acceleration of the earth $\vec{a}_0$

$$\vec{a}_0 = -\frac{GM}{R^2}, \quad \vec{a}_1 = -\frac{GM}{(R - r_0)^2}, \quad \vec{a}_2 = -\frac{GM}{(R + r_0)^2}, \quad (72)$$

where mins sign represents attraction.

Therefore, sea water facing to the moon is faster while sea water in the back of the earth is left behind,as displayed in Figure 5.

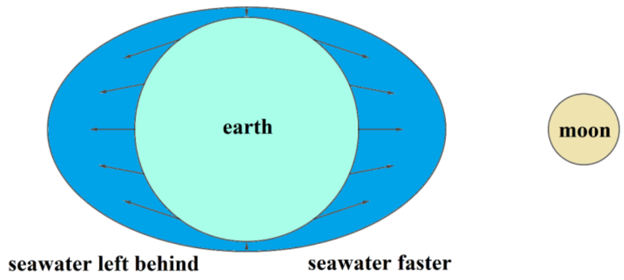

Fig.5 Diagram of the Tidal Force on the Earth

It should be noticed that tidal force is an inertia force and appears only in non-inertial system.

In the frame of general relativity, tidal force is described by the equation of geodesic deviation, which describes the relative motion of two adjacent freefalling objects.

Let x^α, $\tilde{x}^\alpha$ be the coordinate functions of two adjacent geodesics with affine parameter λ. According to the geodesic equation Eq.(72),

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0, \quad (73)$$

$$\ddot{\tilde{x}}^\alpha + \tilde{\Gamma}^\alpha{}_{\beta\gamma}\dot{\tilde{x}}^\beta\dot{\tilde{x}}^\gamma = 0, \quad (74)$$

where $\dot{x}^\alpha = \frac{dx^\alpha}{d\lambda}$, $\tilde{\Gamma}^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma}(\tilde{x})$.

Let $\xi^\alpha(\lambda) = \tilde{x}^\alpha(\lambda) - x^\alpha(\lambda)$. Since these two geodesics are close enough to each other,

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma}(\tilde{x}) = \Gamma^\alpha{}_{\beta\gamma} + \partial_\delta\Gamma^\alpha{}_{\beta\gamma}\xi^\delta. \quad (75)$$

When neglecting higher order term of ξ^α, (74) becomes

$$\ddot{\xi}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{\xi}^\gamma + \Gamma^\alpha{}_{\beta\gamma}\dot{\xi}^\beta\dot{x}^\gamma + \partial_\delta\Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma\xi^\delta = 0. \quad (76)$$

Though Eq.(76) is a second-order differential equation of ξ^α, we need the derivative to be covariant. The directional covariant derivative is defined to be

$$\frac{D\xi^\alpha}{D\lambda} = \dot{x}^\beta\nabla_\beta\xi^\alpha. \quad (77)$$

Then, the second-order covariant derivative of ξ^α can be written as

$$\begin{aligned}\frac{D^2\xi^\alpha}{D\lambda^2} &= \dot{x}^\gamma\nabla_\gamma(\dot{x}^\beta\nabla_\beta\xi^\alpha)\\ &= \frac{d}{d\lambda}\left(\dot{\xi}^\alpha + \dot{x}^\beta\Gamma^\alpha{}_{\beta\delta}\xi^\delta\right) + \dot{x}^\gamma\Gamma^\alpha{}_{\gamma\delta}\dot{x}^\beta\partial_\beta\xi^\delta + \dot{x}^\gamma\Gamma^\alpha{}_{\gamma\delta}\dot{x}^\beta\Gamma^\delta{}_{\beta\mu}\xi^\mu \quad (78)\\ &= \ddot{\xi}^\alpha - \Gamma^\beta{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\Gamma^\alpha{}_{\beta\delta} + \dot{x}^\beta\partial_\mu\Gamma^\alpha{}_{\beta\delta}\dot{x}^\mu\xi^\delta + \dot{x}^\beta\Gamma^\alpha{}_{\beta\delta}\dot{\xi}^\delta\\ &\quad + \dot{x}^\gamma\Gamma^\alpha{}_{\gamma\delta}\dot{x}^\beta\partial_\beta\xi^\delta + \dot{x}^\gamma\Gamma^\alpha{}_{\gamma\delta}\dot{x}^\beta\Gamma^\delta{}_{\beta\mu}\xi^\mu.\end{aligned}$$

Insert Eq.(75) into Eq.(76) and use $\frac{D^2\xi^\alpha}{D\lambda^2}$ instead of $\ddot{\xi}^\alpha$ By simplification, recall the definition of Riemann curvature (10), we get the equation of geodesic deviation:

$$\frac{D^2\xi^\alpha}{D\lambda^2} = R^\alpha{}_{\mu\delta\beta}\dot{x}^\beta\dot{x}^\mu\xi^\delta. \quad (79)$$

$\frac{D^2\xi^\alpha}{D\lambda^2}$ is called the 4-acceleration caused by the curved spacetime, which is the effect of tidal force in common sense. As we can see in Eq.(79), the tidal force is proportional to the Riemann curvature tensor $R^{\alpha}{}_{\mu\delta\beta}$. In our frame, the non-vanishing components of Riemann curvature tensor are

$$\begin{cases} R^{t}{}_{rrt} = 2R^{\theta}{}_{r\theta r} = 2R^{\phi}{}_{r\phi r} = \dfrac{r_s}{r^2(r_s - r)}, \\ 2R^{t}{}_{\theta\theta t} = 2R^{r}{}_{\theta\theta r} = R^{\phi}{}_{\theta\phi\theta} = \dfrac{r_s}{r}, \\ 2R^{t}{}_{\phi\phi t} = 2R^{r}{}_{\phi\phi r} = -R^{\theta}{}_{\phi\phi\theta} = \dfrac{r_s \sin^2\theta}{r}, \\ R^{r}{}_{trt} = -2R^{\theta}{}_{t\theta t} = -2R^{\phi}{}_{t\phi t} = \dfrac{r_s(r_s - r)}{r^4}. \end{cases} \tag{80}$$

Therefore, tidal force in any direction tends to infinity when r tends to 0.

For example, when the mass of the black hole equals three times of the mass of the sun, for an astronaut with a weight of 75 kg and height of 8 m who is 30 km away from the center of the black hole, the radial tidal force pressure amounts about 10^5 times the atmospheric pressure while the maximal pressure a human body can withstand is 100 times the atmospheric pressure. Consequently, the astronaut will be torn apart easily by the tidal force when getting close to the black hole even before getting into the event horizon which is 9 km away from the center (3).

Not only human bodies, most of celestial body cannot withstand the spaghettification effect or the strong tidal force around the black hole. They would be torn up into pieces when getting close to the black hole. Most of the pieces would finally fall into the singularity like a spaghetti. At the beginning, these pieces surround the black hole in some orbits outside the event horizon with almost speed of light, forming the accretion disk as we can see. Curved light would help us to see the whole accretion disk even behind the black hole.

Figure 6 is a picture from the famous film, *Interstellar* (Christopher Nolan, 2014). This black hole is called *Gargantua* (5). The back of the accretion disk is shown in the upper and lower region outside the spherical dark region, which is the image of event horizon including the back of the event horizon. Another important phenomenon we should notice is that the right-hand side part of the accretion disk looks brighter to us. This is a result of *Doppler effect*. Matters in the right-hand side are moving towards us, which makes the frequency of light larger.

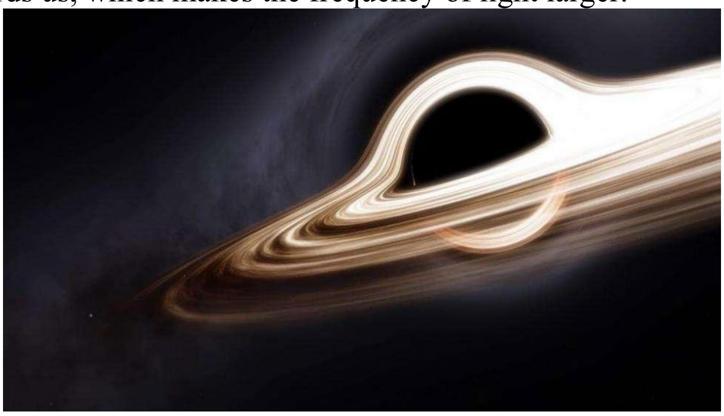

Fig.6 Gargantua from the Movie Interstellar

4. Summary and Comment

In this paper, we discuss various effects that caused by Schwarzschild metric.

In the introduction section, we introduce some basic definitions and consequences of Riemann geometry, like Christoffel symbols, geodesic equation and Riemann curvature tensor. Einstein field equation comes along with the idea that "matter curves spacetime". When the spacetime is static and isotropic, we get the Schwarzschild metric.

Next, the coordinate singularity at $r_s=2GM$ leads us to talk about the event horizon where light cone is hidden. To make the motion of light clear, we also discuss lightlike geodesic equation under such metric and find the photon sphere. In the end of the third section, we introduce Penrose diagram, which is an efficient way to understand the motion of light.

The fourth section helps us to understand why a black hole traps surrounding objects while the third section tells us how the black hole destroys them. By geodesic equation, we witness the mightiness of tidal force. We end the fourth section using a picture from the film *Interstellar* (Christopher Nolan, 2014). By this picture, we can see the objects moving around the black hole *Gargantua,* which forms an accretion disk.

Other black holes such as Reissner-Nordstrom black hole and Kerr black hole also provide prefund viewpoints to study. It will also be interesting to consider other cosmic consequence these black holes result in, like Hawking radiation and black hole jets. Other properties of black holes like the total mass, entropy and surface area also remain various topics to talk about.

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